# Unstable Periodic Orbits and Attractor of the Lorenz Model

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# Unstable periodic orbits and Attractor of the Lorenz Model.

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**Abstract:** Numerical method for detection of unstable periodic orbits on attractors of nonlinear dynamical systems is proposed. This method requires the similar techniques as the data assimilation does. This fact facilitates its implementation for geophysical models.

Some low-period orbits of the Lorenz model have been calculated explicitely. The orbits encoding and application of symbolic dynamics is used to classify and identify the detected orbits and find the whole set of fundamental cycles. Application of the cycle expansion theory to the fundamental cycles set allows to approximate some attractor characteristics difficult to calculate directly.

**Key-words:** Unstable periodic orbits, attractor, nonlinear dynamical system, Lorenz model, geophysical models.

*(Résumé : tsvp)*

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# Orbites périodiques instables et l'attracteur du modèle de Lorenz.

**Résumé :** Une méthode numérique de détection des orbites périodiques instables sur l'attracteur d'un système dynamique non linéaire est proposée. Cette méthode utilise les même techniques que l'assimilation de données. Ce fait simplifie son implémentation aux modèles géophysiques.

Quelques orbites périodiques du modèle de Lorenz sont calculées explicitement. Le codage de ces orbites et l'application de la dynamique symbolique est utilisé pour leur identification afin de trouver tout l'ensemble fondamental des cycles. L'application de la théorie des expansion des cycles permet d'approcher les caractéristiques de l'attracteur qui sont difficiles à calculer directement.

**Mots-clé :** Orbites périodiques instables, attracteur, système dynamique non linéaire, modèle de Lorenz, modèle géophysique.

# 1 Introduction.

Numerous recent studies have been focussed on the properties of chaotic solutions generated by non-linear systems. One of the major fields where this kind of solutions is extremely important is the atmospheric and oceanic dynamics. The importance is based on the necessity to deliver the weather forecasts and the limited time of deterministic forecasting of chaotic systems. The success of long-range forecasting depends on the understanding of the sources and nature of the variability of the system beyond the timescale of deterministic prediction.

Low frequency variability of nonlinear dynamics of the atmosphere has long been discussed. After discovery of deterministic chaos [Lorenz, 1963] the interpretation of the atmospheric and oceanic circulations, their variability and predictability have been developed by tools of the dynamical systems theory.

One of the approaches to apply the dynamical system theory is to extract local characteristics of the strange attractor. It is well known that the behaviour of a solution can differ a lot on different parts of the attractor. The attractor inhomogeneity leads to the phase spatial variability in the predictability time scale. This variability can result, for example, from the proximity of trajectories to unstable stationary points and their stable and unstable manifolds. This idea has been quantified in various ways by using local phase space information.

The study of this kind of inhomogeneity for atmospheric and oceanic dynamics has been started by the work of [Charney and De Vore, 1979], who showed that a system of atmospheric flows in a rotating channel with a spatially inhomogeneous forcing can have several unstable equilibria.

The theory of multiple equilibria [Charney and De Vore, 1979] of the atmospheric circulation developed in order to explain and classify quasi stationary atmospheric regimes can be considered as a good example of the application of dynamical systems theory to the climate. The question of multiple weather regimes has been addressed in numerous analytical and observational studies since half of century ago.

The earliest notions of their classifications relate to high and low circulation indices [Rossby, 1939], zonal and meridional circulation indices [Blinova, 1943], blocked and zonal flows [Berggren *et al.*, 1949], [Rex, 1950]. These empirical classifications have been revisited and their properties have been explained basing on the hypothesis that when a system passes a state close to one of multiple equilibria, quasi stationary atmospheric regime results.

Each hyperbolically unstable equilibrium forms a region in the phase space where the system trajectories slow down before being ejected along an unstable manifold. The reasonability of this hypothesis has been confirmed by the comparison of the results of cluster analysis of the solution of the barotropic model with its equilibria [Mo and Ghil, 1988]. The analysis shows closeness of the clusters centroids with the equilibria.

Using a truncated barotropic model of the atmosphere [Legras and Ghil, 1985] demonstrated that recurrent quasi stationary states occur in the vicinity of unstable stationary points in the phase space of the model. The lifetime of this regime depends on a particular trajectory in the phase space [Mo and Ghil, 1987], however the mean regimes lifetime related to the stability characteristics of the adjacent stationary point [Dymnikov *et al.*, 1990]. This fact allows to obtain a priori estimates of the regimes lifetime. The climatic average of the barotropic model solution has been analysed in [Dymnikov and Kazantsev, 1993]. It is shown that this average can be developed by the set of equilibria with a rather good accuracy.

Essential results are recently obtained also for theoretical investigations of the phase space structures. Theorems for the existence and uniqueness of a solution, existence of the finite-dimensional attractor for geophysical models have been proved in [Temam, 1988], [Dymnikov and Filatov, 1990], [Bernier, 1994]). Analysis of the number of equilibria in the phase space of the model has been performed in [Filatov, 1992]. Mathematical analysis of climatic processes can be found in [Dymnikov and Filatov, 1996]. These works provide us with a good theoretical basis for the forthcoming researches.

However, the applications of this theory to the climatic models analysis possesses one principal "shortcoming". Despite the fact that there exists multiple equilibria, their vicinities cover a very small part of the attractor only. Thus, the model spends much time out of these vicinities where its behaviour can not be explained by means of this theory.

So far the analysis of equilibrium points of chaotic models has been so fruitful, one can try to study another kind of "particular solutions" of a nonlinear system. This kind of solution is the periodic orbits (or limit cycles) which also can exist on the attractor of the system. As well as equilibria discussed above, they can be as stables and unstables. However in practice, unstable equilibria and unstable periodic orbits only call a particular attention in the studies chaotic system, because any stable solution used to form a regular attractor with no chaos.

Similarly to quasi stationary regimes which are explained by the motion near unstable equilibrium, we can speak of quasi periodic regimes, or intermittent appearances of oscillatory modes, which may be explained by the motion near unstable limit cycle.

The periodic orbits for geophysical model have attracted some interest already ([Itoh and Kimoto, 1996],[Jiang *et al.*, 1995],[Dymnikov *et al.*, ]). However all these papers address the stable periodic orbits on the bifurcation diagram and the transition to chaos.

There exists the principal difference between stationary solution and periodic orbits. The number of equilibria is usually finite for a geophysical model. Moreover, it is always finite for a finite dimensional approximation of a model. So far the nonlinearity is quadratic, the number of stationary points is bounded by $2^N$ where $N$ is number of degrees of freedom of the discretisation. So one can hope to find all equilibria of the model. However the number of periodic orbits is usually infinite even for discretised model. To say more, one of definitions of the chaotic system [Devaney, 1987] used the density of periodic orbits on some set $V$ to define the latter as a set specificallywhere the behaviour of the system is said to be chaotic. Following this definition, if we speak of a chaotic system, its periodic orbits are dense in some $V$, hence they are of infinite number. This is the case of Lorenz system [Lorenz, 1963], which is composed of 3 ordinary differential equations only.

Despite we can not find all the periodic orbits, we can find and analyse some of them, with lowest period. The situation is similar in some sense to the analysis of equilibria where we can not prove we found all of them and we have to analyse equilibria we have found only.

One can say that for some purposes only a limited number of low-period orbits may be sufficient. This point of view is argued in [Hunt and Ott, 1996b], [Hunt and Ott, 1996a]. However, this conclusion does not hold generally and some applications may require long-period orbits also [Zoldi and Greenside, 1997a]. In this case it may be possible to apply the cycles expansion theory [Arutso *et al.*, 1990a], [Arutso *et al.*, 1990b] in order to manage all the periodic orbits set.

To treat periodic orbits of a geophysical system we need first to detect some of them numerically. This requires an efficient algorithm of the unstable periodic orbit search, applicable to geophysical models.

To develop such an algorithm and to analyse its properties and difficulties of its application we use first an extremely simple chaotic model with a strange attractor, namely the Lorenz model [Lorenz, 1963]. This model has been carefully investigated for a last three decades, we know much about its attractor.

In the first section of the paper the method of unstable periodic orbits search is proposed. The second section is devoted to the analysis of their properties related to the predictability and attractor of the model.

## 2 Instable periodic orbits of the Lorenz model.

The Lorenz model [Lorenz, 1963] writes

$$\begin{aligned} \frac{dx}{dt} &= \sigma y - \sigma x \\ \frac{dy}{dt} &= -xz + rx - y \\ \frac{dz}{dt} &= xy - bz \end{aligned} \quad \text{with parameters} \quad \begin{aligned} \sigma &= 16, \\ b &= 4, \\ r &= 45.92. \end{aligned} \tag{1}$$

This model with the parameters given possesses 3 unstable stationary points:

$$(0,0,0), (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$$

There exists a strange attractor in the model phase space which has a dimension about 2.06.

The advantage of this model is its simplicity. Its attractor, bifurcation diagram, stationary points and periodic orbits have been studied carefully. Much information about this model has been collected in the book [Sparrow, 1982]. In particular, there has been proposed to use the Newton method to locate unstable periodic orbits on the attractor.

However, the initial conditions for this method must be chosen in a rather close vicinity of a periodic orbit, otherwise the method diverges. To simplify the choice of the initial point, there has been proposed in [Zoldi and Greenside, 1997b] to use dumped-Newton method. This method differs from the classical Newton method by the fraction $\alpha \leq 1$ of a Newton correction $\delta x$ is added only to update the unknowns, $x \to x + \alpha\delta x$. This method found to be more efficient to find periodic orbits due to less restrictive choice of the initial guess.

However realisation of this method requires $O(N^3)$ operations per iteration due to necessity to calculate the matrix of the Newton process and solve a system of equation with this matrix. This fact limits its use by low-dimensional systems only. In fact, this kind of methods works well for the Lorenz system and even for higher dimensional systems, however the number of variables must not exceed 100.

Several methods based on the stabilisation of an unstable periodic orbits have been proposed also. There has been discussed in [Barreto *et al.*, 1997] the possibility to find a "window" in the parameter range of the model where one of its periodic orbits become stable. However, the possibility to find these kind of "window" depends on the number of positive Lyapunov exponents on the attractor, and if the last is sufficiently large, the search of such "window" may become difficult or impossible. Moreover, it may be difficult to know also to determine whether periodic orbits are subjected to smooth variations only or bifurcations may occur resulting in disappearance of the orbit.

In [Schmelcher and Diakonos, ] a method of stabilisation of orbits based on a universal set of linear transformation, namely special reflections and rotation in space. However, the application of this method to a high dimensional system may not be efficient due to very large number of possible transformation to try.

Our purpose is to develop an efficient method which would be applicable to geophysical systems like atmospheric and oceanic models. First, we can remark the great progress achieved in the variational data assimilation techniques. For many model the data assimilation based on a functional minimisation has been developed and applied. Moreover this minimisation uses gradient-type methods requiring as many operation per iteration as the model does. If we formulate the problem of periodic orbits search as a functional minimisation problem, we can apply the technique similar to variational data assimilation for this model.

Let us consider the functional

$$J(\vec{\xi}, T) = \frac{\|\vec{x}(\vec{\xi}, T) - \vec{\xi}\|^2}{2} \tag{2}$$

where $\vec{x}(\vec{\xi}, T)$ is a solution at time $T$ of the system (1) with initial conditions $\vec{\xi}$. Euclidian norm used here $\|\vec{x}\|^2 = <\vec{x}, \vec{x}> = \sum_{i=1}^{N} x_i^2$.

One can easily see that for every periodic solution of period $T$ the value of this functional $J(\vec{\xi},T)$ is equal to 0 for all orbits originating at any point on this periodic solution. The functional $J$ depends on $N+1$ variables: $N$ components of vector $\vec{\xi}$ and period $T$.

To minimise $J$ we calculate first its gradient at a given point $\vec{\xi}$ and for given $T$ in the $N+1$ variables space.

To simplify notations let us denote:

- the right-hand-side of (1) by $\vec{F}(x)$
- the solution of (1) at time $t$ corresponding to initial condition $\vec{\xi}$ by $x(\vec{t}) = \vec{B}(\vec{\xi},t)$,
- the matrix of the tangent linear model by $G_{i,j}(\vec{\xi},t) = \dfrac{\partial B_i}{\partial \xi_j}$.

Using these notations we get:

$$\nabla J(\vec{\xi},T) = \begin{pmatrix} \dfrac{\partial J}{\partial \xi_1} \\ \cdots \\ \dfrac{\partial J}{\partial \xi_N} \\ \dfrac{\partial J}{\partial T} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \|\vec{B}(\vec{\xi},T)-\vec{\xi}\|^2}{\partial \xi_1} \\ \cdots \\ \dfrac{\partial \|\vec{B}(\vec{\xi},T)-\vec{\xi}\|^2}{\partial \xi_N} \\ \dfrac{\partial \|\vec{B}(\vec{\xi},T)-\vec{\xi}\|^2}{\partial T} \end{pmatrix} \begin{pmatrix} (G^*-I)\times(\vec{B}(\vec{\xi},T)-\vec{\xi})\mid_i, \; i=1,\ldots,N \\ \\ < \dfrac{\partial \vec{B}(\vec{\xi},T)}{\partial T}, \vec{B}(\vec{\xi},T)-\vec{\xi} > \end{pmatrix} \qquad (3)$$

where $G^*$ is the adjoint to $G$ linearised with respect to the orbit $\vec{B}(\vec{\xi},t)$ . For the Lorenz system the adjoint model can easily be written:

$$\begin{aligned} -\frac{dx}{dt} &= -\sigma x + \Big(r - B_3(\vec{\xi},t)\Big)y + B_2(\vec{\xi},t)z \\ -\frac{dy}{dt} &= \sigma x - y + B_1(\vec{\xi},t)z \\ -\frac{dz}{dt} &= B_1(\vec{\xi},t)y - bz \end{aligned} \qquad (4)$$

The last component of the vector $\nabla J(\vec{\xi},T)$ can easily be calculated

$$< \frac{\partial \vec{B}(\vec{\xi},T)}{\partial T}, \vec{B}(\vec{\xi},T)-\vec{\xi} > = < F(\vec{B}(\vec{\xi},T)), \vec{B}(\vec{\xi},T)-\vec{\xi} >$$

So, to calculate $\nabla J(\vec{\xi},T)$ one need to calculate first $\vec{B}(\vec{\xi},T)$, i.e. perform the integration of the direct model (1). It should be noted here that one has to keep all the orbit for further integration of the adjoint model. And after that integrate the adjoint model (4) with initial conditions

$$\vec{x}\mid_{t=T} = \vec{B}(\vec{\xi},T) - \vec{\xi}$$

in order to obtain the product $G^*(\vec{B}(\vec{\xi},T)-\vec{\xi})$. The first $N$ components of the $\nabla J$ are calculated easily as $G^*(\vec{B}(\vec{\xi},T)-\vec{\xi}) - (\vec{B}(\vec{\xi},T)-\vec{\xi})$.

The gradient of $J(\vec{\xi_n},T_n)$ allows us to perform the iterational descent from the point $\vec{\xi_n},T_n$ to the point $\vec{\xi}_{n+1},T_{n+1}$. The initial point of iterations $\vec{\xi_0},T_0$ can be chosen as arbitrary point on the attractor of the model. For example, one can integrate the model for an arbitrary time and take the result of the integration as the starting point for iterations. There exist 3 possibility of the process development:

- the process diverge (does not converge after some fixed number of iterations),
- the process converge to a minimum with some non-zero $J$,

- the process converge to a solution already found,
- the process converge to a new solution.

Only fourth item is considered as good, the first three are ignored.

The descent procedure used here is the truncated Newton method developed by S.G. Nash [Nash, 1984]. The truncated-Newton method is preconditioned by a limited-memory quasi-Newton method with a further diagonal scaling.

Direct (1) and adjoint (4) models have been discretised in time with time step 0.002. 410 periodic orbit of the Lorenz system have been found using the method presented above. The precision we require to stop the iterations is $J < 10^{-10}$. To achieve this precision some 20-70 iterations required usually. We used 200 iterations as upper limit of iterations, if the number of iterations become larger, the process is considered as divergent and the result is ignored.

The Lorenz system possesses an obvious symmetry: the transformation $x \to -x, y \to -y, z \to z$ does not change the system. Hence for any asymmetrical periodic orbit, the orbit obtained as the result of this transformation is also periodic. And of course, among the periodic orbits there can exist symmetrical orbits with respect to this transformation.

In figures fig.1 one can see two examples: symmetrical orbit with period $T = 0.9149$ and two asymmetrical ones with $T = 1.7857$.

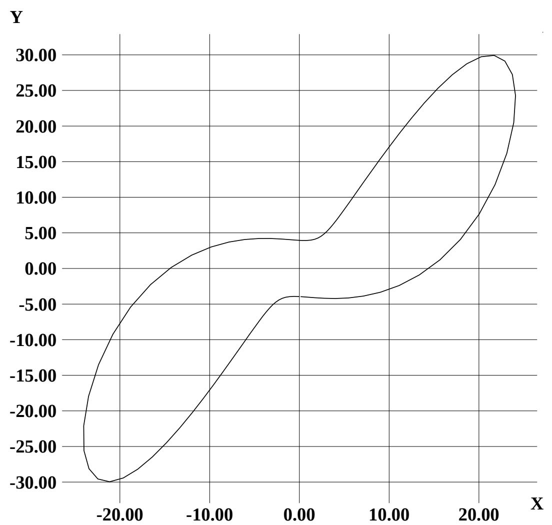


Figure 1A. Projection on the plan $x-y$ of the symmetrical periodic orbit $T = 0.9149$

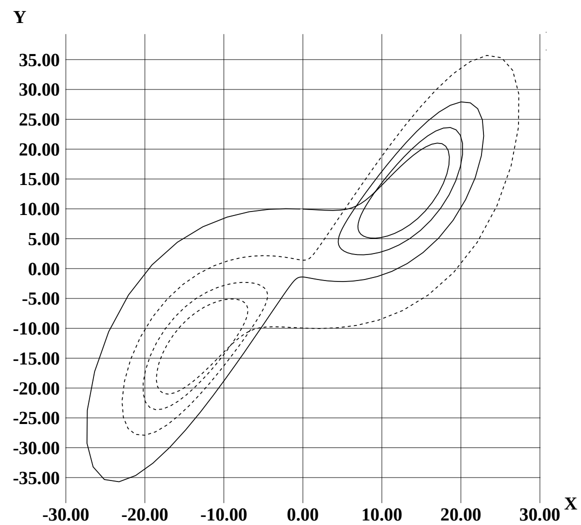


Figure 1B. Projection on the plan $x-y$ of two asymmetrical periodic orbits $T = 1.7857$

In order to provide a numerical evidence of the density of the periodic orbits on the attractor, the Poincaré section by planes $x = 0$ and $z = r - 1$ of all the periodic orbits is shown in the figure fig.2. One can easily see periodic orbits cover the attractor region.

These sections are used also for the identification of the periodic orbits found. So far we can use any point on the orbit as its initial condition, this results in "non uniqueness" and provide difficulties with its identification.

To be capable to distinguish a new solution from already found ones, we look for a point on this solution for which $x = 0$, i.e. the points where the orbit intersect the plane $x = 0$ fig.2A. There exists at least 2 such intersections, so we choose one for which $y$-coordinate has the lowest possible value. This point is obviously unique. We keep this point as the initial point for the orbit and to compare different orbits it becomes sufficient to compare their initial points.

The second plot we shall use further for the orbits encoding.

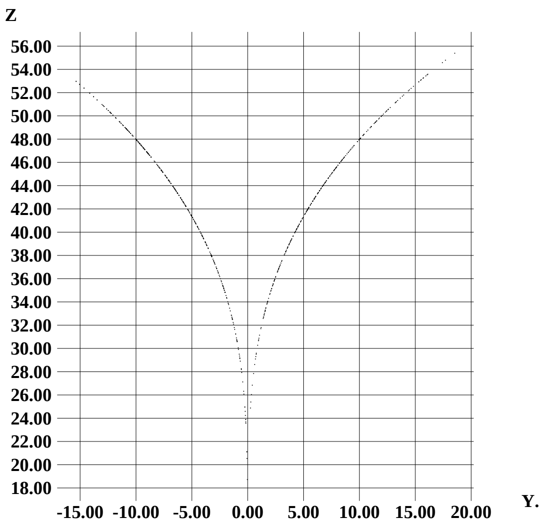


Figure 2A. Poincaré section by plane $x = 0$

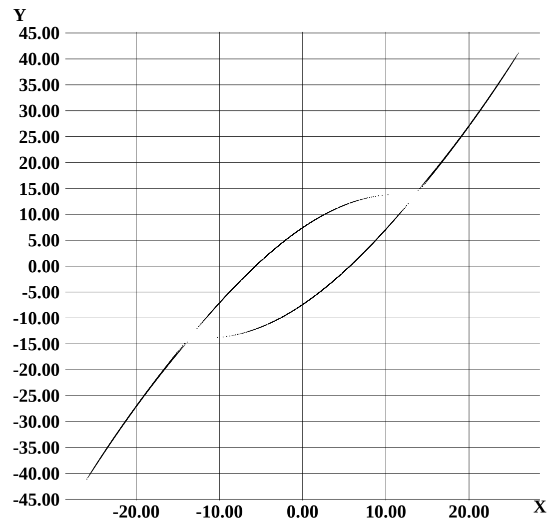


Figure 2B. Poincaré section by plane $z = r - 1$

One of interesting features of periodic orbits of the Lorenz model is their distribution on periods. One can see on fig.3 that periodic orbits are concentrated in groups with periods near multiples of some elementary period $T \sim k \times T_0$. The value of this period can easily be calculated as $T_0 = \dfrac{2\pi}{\sqrt{b(r^* + \sigma)}}$ where $r^*$ is the value of parameter $r$ at the bifurcation point $r^* = \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$. The value of $T_0$ for the parameters (1) is equal to $T_0 = 0.447$. It should be noted here that there is no periodic orbits with period $T_0$, this orbit has disappeared in the subcritical Hopf bifurcation at $r = r^*$ when non-zero stationary points loose its stability.

The regularity in fig.3 relates to the bifurcation chains the orbits have been generated in, i.e. doubling period cascade and symmetry breaking bifurcations. This regularity is the best viewed in the Nusselt number plot versus the solution period. The Nusselt number is a dimensionless spatially averaged measure of the vertical heat transport of a convection cell. For the Lorenz system this transport is given by

$$N = 1 + \frac{2}{br} xy.$$

In fig.3A averaged Nusselt number $\bar{N} = \frac{1}{T} \int_0^T N(t)dt$ is plotted for each periodic orbit found. One can easily see that each point lies at the intersection of two hypothetic curves: almost straight line which is almost orthogonal to the $T$ axis, and a hyperbolic-type curve. The strait lines represent the groups of orbits with periods close to $T_k \sim k \times T_0$. Each hyperbolic curve starts at some low period orbit and follow all orbits appeared in the certain type of bifurcations.

The set of hyperbolic curves begins with the "upper" hyperbole originating at the orbit with lowest period $T = 0.91$ Each other hyperbole originates at the orbit with higher period and, in some sense, is parallel to the first one. One can suppose that these hyperboles could be continued to infinite period and so they contain infinite number of orbits. The upper hyperbole represents the sequence of the most asymmetric orbits which make one turn around one non-zero stationary point and $k$ turns around the another one. It is clear that when $T \to \infty$ the Nusselt number tends to the Nusselt number of the stationary point. So the Nusselt number of the stationary point is the horizontal asymptote of this hyperbole.

Another type of subset is the set of groups of periodic orbits with close periods which form strait almost vertical lines in fig.3A. If we denote that the first two corresponds to lowest periods, they are composed of one point each and so they can not be considered as lines. However, the third line is composed of two points, the fourth one — of 3 etc.

One can distinguish the same type of regularity in the fig.3B where the $z$ coordinate of initial conditions is plotted versus the period. We remind that the initial conditions have been chosen as points where the orbit intersect the plane $x = 0$ (fig.2A) and the $y$-coordinate has the lowest possible value. The distribution in $z$ is well delimited by two hyperbolic curves. "Upper" hyperbole correspond to the orbit which makes $k$ turns around the equilibrium with negative $x$ and $y$ and the "lower" one corresponds to the orbit which makes $k$ turns around the equilibrium with positive $x$ and $y$ . One can see that in this case the groups with close periods have more complicate structure than simple lines.

There are more points on the second plot due to the fact that any couple of asymmetrical orbits has the same Nusselt number and so this couple is presented by one point in fig.3A, while their $z$ coordinates are different and provide two points in fig.3B.

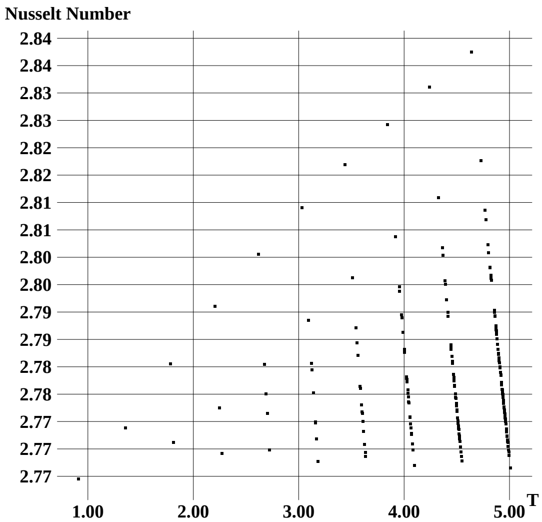


Figure 3A. Averaged Nusselt number vs T.

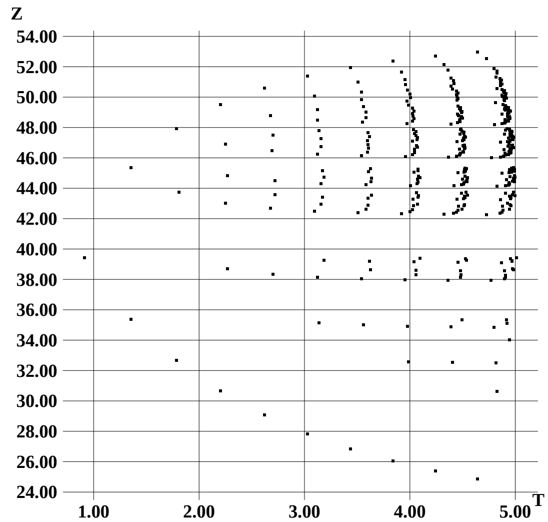


Figure 3B. $z$ coordinate of initial conditions vs T.

To perform the extrapolation of periodic orbits up to infinite period or perform the cycle expansion, we have to identify them first. This identification will help us also to calculate the number of orbits in each group. It can be performed by the symbolic dynamics analysis of periodic orbits [Hao and Liu, 1997], [Bedford *et al.*, 1991]. To each orbit a symbolic sequence is associated. Analysis of the Lorenz model carried out in [Sparrow, 1982] that this coding is possible for some range of the parameter $r$ and at most one cycle corresponds to each symbol sequence. Following [Franceschini *et al.*, 1993], we use quasi one-dimensional set in the Poincaré section of the Lorenz attractor by $z = r - 1$ plane presented in fig.2B. It is noticed that upward intersections of the orbits may be directly parametrised by $x$ coordinate. So far upward intersections of periodic orbits occur at $\mid x \mid > \sqrt{b(r-1)}$ only, we denote by the symbol "A" each intersection occured at $x > \sqrt{b(r-1)}$ and by "B" at $x < -\sqrt{b(r-1)}$. Thus each orbit is encoded by the sequence of symbols "A" and "B".

The number of symbols in sequence is the same for the same group. The orbits showed in fig.1A and B are encodes by "$AB$" and "$ABBB$" respectively. As it follows directly from the nature of periodic orbits, the sequences "$ABBB$" and "$BBAB$" represent the same orbit with different initial conditions on it. So we do not need to consider these sequences as different. Similarly, "$ABAB$" represents the same orbit as "$AB$" but repeated two times. Taking into account that any orbit intersect the plane $z = r - 1$ at least twice, we must consider only the sequences

$$\Sigma^{(k)} = (\sigma_1, \sigma_2, \ldots, \sigma_k), \forall i \ \sigma_i \in \{A, B\} \tag{5}$$

which satisfy:

- $\exists i$ such that $\sigma_i\sigma_{i+1} = AB$,
- $\forall i, j\ \Sigma^{(i)}\Sigma^{(j)} = \Sigma^{(j)}\Sigma^{(i)} = \Sigma^{(i+j)}$,
- $\nexists i \neq k$ such that $\Sigma^{(k)} = \left(S^{(i)}\right)^{k/i}$.

The product of two sequences used here is the sequence composed of all their elements: $\Sigma^{(i)}S^{(j)} = (\sigma_1, \ldots, \sigma_i, s_1, \ldots, s_j)$.

The number of possible sequences of length $k$ represents the number of periodic orbits appeared after $k$ bifurcations. Their periods are close to $k \times T_0$ and, hence, this number is equal to the the number of points in each group of orbits in fig.3B with such periods.

This number can be estimated by the formula [Chassaing, 1997], [Arutso *et al.*, 1990a],

$$N^{(k)} = \frac{1}{k}\Bigg(\sum_{i:\mathrm{mod}(k,i)=0} \mu(i)2^{k/i}\Bigg) \tag{6}$$

where

$$\mu(i) = \begin{cases} 0 \text{ if } i = i_1^{\alpha_1} i_2^{\alpha_2} \ldots i_p^{\alpha_p} \text{ and } \exists j \text{ such that } \alpha_j > 1 \\ (-1)^{p+1} \text{ if } \forall j\ \alpha_j = 1 \end{cases} \tag{7}$$

| Group number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Length of sequences | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Period $T$ | 0.9 | 1.3 | 1.8 | 2.2 | 2.7 | 3.1 | 3.6 | 4.0 | 4.5 | 4.9 | 5.4 |
| Number of orbits | 1 | 2 | 3 | 6 | 9 | 18 | 30 | 56 | 99 | 186 | 335 |

Encoding of periodic orbits by symbolic sequences provides us with the following information and facilitate the study of the attractor structure of the model. First, we can easily see whether all the solution with periods $T$ less than some $\bar{T}$ have been found or not. The number of possible sequences of length $k$ represents the number of periodic orbits in $k$-th group, so the number of found orbits in all groups with periods less than $\bar{T}$ should be counted and compared with the necessary values obtained from (6). This analysis shows that we have found all 410 periodic orbits with period lower than 5, one can easily see that 410 is equal to the sum of all possible symbol sequences of length 10 or less.

Second, we can easily get a rather good initial guess for the descent procedure for any solution which has not been found already and so find it. To obtain the initial guess one can analyse the plots of the initial conditions as the fig.3B and similar figures for other coordinates. The regularity of these plots allows and locate approximately the initial conditions for the missed orbits and their periods. Starting the descent procedure from the approximated initial conditions, we can find the missing orbit.

Using the representation of periodic orbits by symbolic sequences one can obtain easily some characteristics of the attractor. For example we can easily calculate the topological entropy of the Lorenz attractor discussed in [Auerbach *et al.*, 1987], [Franceschini *et al.*, 1993] using the formula (6)

$$K_0 = \lim_{k\to\infty} \frac{\ln N^{(k)}}{k} = \ln 2$$

And the third point, more attractor characteristics, which are difficult to calculate directly can be obtained from the cycle expansion methods. This theory, proposed in [Cvitanovic, 1988], shows that only a subset of *fundamental cycles*, which includes the minimal number of cycles only, is important,because they are sufficient for a correct description of the topology of the attractor. Any other cycle can be approximated by elements of the fundamental set. Thus, despite the number of periodic orbits is infinite on the attractor, we do not need to look for all of them to understand their properties and the properties of the attractor.

# 3 Periodic orbits, predictability and attractor.

One of the interesting aspect of periodic orbits is their instability characteristics. As it has been mentioned in the introduction, the study of the stationary points of the system of the barotropic atmosphere and the corresponding quasi-stationary regimes of atmospheric circulation brought fruitful results in the domain of analysis and a priori estimates of the lifetime of such regimes.

In [Dymnikov and Kazantsev, 1993] the theory of instability indices for stationary points has been applied to estimate the mean time the trajectory spend in an equilibrium vicinity. These theory have been applied even for real blocking-type regimes of the circulation over north Atlantic and Europe in [Dymnikov *et al.*, 1990]. There has been shown the relationship between the instability index and the lifetime of blockings. This theory is based on the supposition that a quasi-stationary regime arises when the trajectory approaches an unstable equilibrium through its stable manifold and withdraw from it through the unstable one. So it is natural to deduce that the mean duration of the trajectory's stay in the vicinity of an equilibrium is proportional to the characteristics of the unstable manifold of the equilibrium, as well as the number of intrusions of the trajectory into the vicinity is proportional to the characteristics of its stable manifold. Hence the mean duration of the circulation regime becomes proportional to the instability characteristics and the frequency of occurrence of the regime to the stability characteristics of the corresponding equilibrium.

In this model study we try to develop this idea and apply it to the periodic solutions, and consequently to define quasi-periodic regimes. The observational evidence of the existence of such kind of regimes, or waves, can be found in any tutorial for a physical systems like atmosphere or ocean. However, the question is open whether they can be explained by the presence of an unstable periodic orbit near them.

To measure the instability of the periodic orbit we use the sum of positive Lyapunov exponents as a characteristic of the divergence rate of nearby trajectories close to the periodic one. They are based on the eigenvalues of the operator of the tangent linear model linearised around the periodic orbit. For the Lorenz model (1) one can simply get the tangent linear model as

$$\frac{d\vec{x'}}{dt} = A(\vec{B})\vec{x'}, \text{ where } \vec{x'} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \text{ and } A\Big(\vec{B}(\vec{\xi},t)\Big) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - B_3(\vec{\xi},t) & -1 & -B_1(\vec{\xi},t) \\ B_2(\vec{\xi},t) & B_1(\vec{\xi},t) & -b \end{pmatrix} \quad (8)$$

where $\vec{B}(\vec{\xi},t)$ is a point on the periodic orbit at moment $t$ originating at point $\vec{\xi}$.

The sum of positive Lyapunov exponents can be calculated as the

$$K = \lim_{t\to\infty} \frac{1}{2t} \ln \sum_{i:\lambda_i>0} \lambda_i(G_t^* \times G_t) \text{ where } G_t = \exp \int_0^t A(\vec{\xi},t)dt \sim \prod_{t=0}^{t} (I + \tau A(\vec{\xi},t)) \quad (9)$$

The limit appeared in the definition of the Lyapunov exponents causes the difficulties of their calculation due to necessity to perform very long time integration. However, the use of periodic orbits can simplify this problem because for any of them this limit can be calculated in finite time thanks to periodicity. So far we know the same orbit will be repeated all the time, we replace the limit $\lim_{t\to\infty}$ by $\lim_{n\to\infty}$, where $n$ is the number of repetition of the orbit. So

$$K = \lim_{n\to\infty} \frac{1}{2nT} \ln \sum_{i:\lambda_i>0} \lambda_i\Big((G_T^*)^n \times (G_T)^n\Big) = \frac{1}{T} \ln \sum_{i:\lambda_i>0} |\lambda_i(G_T)| \quad (10)$$

Thus, the Lyapunov exponents are related to the Floquet multipliers and can be calculated within one period integration.

We shall use the hypothesis that a quasiperiodic regime arises when the trajectory approaches to an unstable periodic orbit. We suppose also that the mean duration of this regime is related to the instability characteristics of the orbit while the frequency of occurences of the particular quasi periodic regime is related to the stability characteristics of the corresponding orbit.

In order to verify this hypothesis, the mean time spent in the vicinity is estimated directly. We define $\varepsilon$-vicinity of the periodic orbit as a torus centred on the orbit with circular section of radius $\varepsilon$. A set of integration of orbits close to periodic one is performed. In each integration a small arbitrary perturbation to the periodic orbit is added and the averaged time the perturbed trajectory remains in the vicinity is measured. Despite the Lyapunov exponents by definition explain the divergence rate of infinitesimal perturbation, the relationship between them and the time spent in the vicinity is not evident due to finiteness of a real vicinity and due to possible non-symmetry of the $G$ operator which leads to the "super-exponential error growth" [Nicolis *et al.*, 1995]. However, the Lyapunov exponents remain good estimates of the mean time spent in a finite vicinity for Lorenz model. In fig.4A one can see the relationship between the mean time spent in vicinity of the periodic orbit and its positive Lyapunov exponent. The radius of the vicinity 0.5, the norm of the initial perturbation $10^{-5}$. One can see the correlation between these values remains high $(0.96 \pm 0.05)$ for as large vicinity as 0.5

To estimate the frequency of occurences of a particular quasi periodic regime directly we should distinguish the events when the trajectory passes close to the periodic orbit. However, it is not possible to cover all periodic orbits by non intersecting vicinities and count the number of entrances into the one of them due to density of periodic orbits on the attractor. Even if we work with a limited number of low-period orbits, the vicinity radius has to be chosen very small. This leads to a very long time integration to obtain statistically significant number of entrances in each of them.

Following [Eckhard and Ott, 1994] we address again the symbolic sequences discussed above. A long trajectory on the attractor was run for 2 millions crossing of the plane $z = r - 1$. Similarly to encoding of periodic orbits, each upward crossing the plane by this trajectory was encoded by a symbol "A" or "B" for $x > \sqrt{b(r-1)}$ and $x < -\sqrt{b(r-1)}$ respectively. In the long symbolic sequence of 2 millions symbols obtained in this way, we look for subsequences of a fixed length which correspond to one of periodic orbits. It should be noted that there exists $n$ sequences of length $n$ which correspond to the same orbit obtained by the transposition described by the second item of (5), so we must not distinguish them. In the same time there can exist subsequences which can not be identified with a periodic orbit. These sequences are prohibited by either the first or the third items of (5).

So, any part of trajectory encoded by the sequence of length $n$ can which corresponds to the periodic orbit is considered as close to this orbit. In fact, the trajectory part is not obligatory close to the periodic orbit in the sense of some norm in the phase space, it only "follows" the periodic orbit intersecting the plane in the same regions. In fig.5B one can see the comparison of the trajectory segment encoded by the symbol sequence of length 9 "$BBBAABAAA$" and the periodic orbit with period $T = 4.045077$ with the same encoding.

However, the probability to find a sequence corresponding to the periodic orbit in the long symbol sequence and the Floquet multiplier of the corresponding periodic orbit are in a good agreement as demonstrated in fig.4B. The correlation between is about 98%. Each of five sets corresponds to sequences of lengths from 7 to 11. So far shorter sequences are composed of lower number of periodic orbits, the relative probability of their realisation is higher. Thus the sets which represent shorter sequences are situated higher in fig.4B.

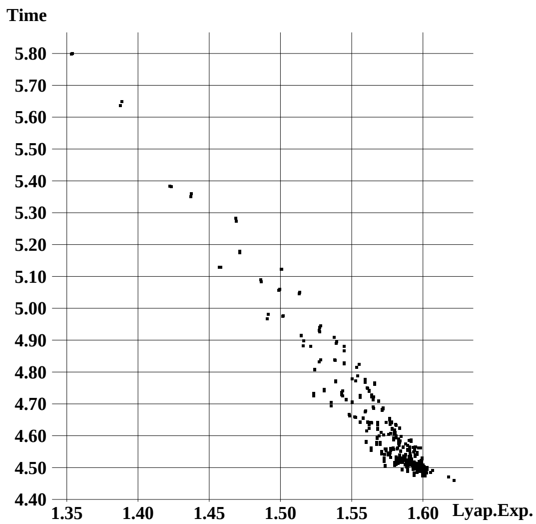


Figure 4A. Mean time spent in vicinity of the periodic orbit versus its Lyapunov exponent.

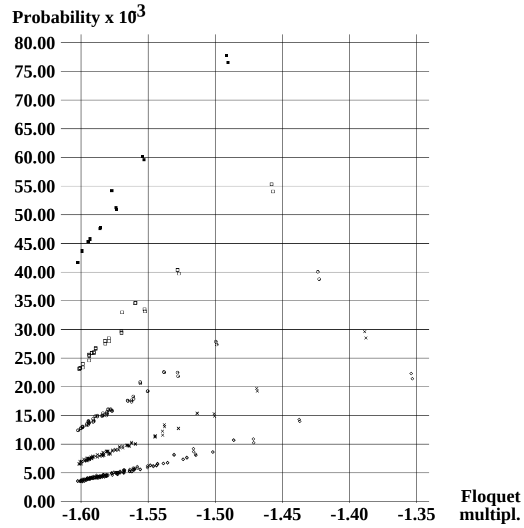


Figure 4B. Frequency of occurences of a symbol sequence of length from 7 to 11 identical to the periodic one versus the Floquet multiplier of the corresponding periodic orbit.

The idea to approximate the chaotic attractor properties by means of unstable periodic orbits is not new. The important role played by periodic orbits was noted already by H. Poincaré (1892) et E. Hopf (1942). This interest is reappeared in modern studies. Possibility of studying the strange attractors of dynamical systems by means of periodic orbits is discussed in [Auerbach *et al.*, 1987], the cycle expansion formalism and its convergence is analysed in [Arutso *et al.*, 1990a]. One can find some applications of this formalism in [Eckhard and Ott, 1994], [Franceschini *et al.*, 1993]. An analysis of periodic orbits for Lorenz model can be found in [Sparrow, 1982].

Unstable periodic orbits have been found for higher dimensional systems (Kuramoto-Sivashinsky equation with N=100) [Zoldi and Greenside, 1997b]. The first steps have been performed to distinguish periodic orbits in geophysical systems [Jiang *et al.*, 1995], [Wang and Fang, 1996], [Itoh and Kimoto, 1996].

In this paper we try use low-period orbits to approximate such simple attractor characteristics as the average of the solution and its moments:

$$M_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t x^i dt. \tag{11}$$

The moments can be calculated directly form the long orbit. However, a long integration of the model must be performed to obtain a sufficient accuracy due to slow convergence of the limit. To avoid the necessity of the long model integration we can use periodic orbits to approximate the moments. Approximation of the first moment of the solution, i.e. its average, has been discussed already [Hunt and Ott, 1996a],[Hunt and Ott, 1996b]. However, approximation of higher moments may be important also in applications.

Approximation of moments by moments of periodic orbits is performed as weighted sums, with weights equal to the inverses of Lyapunov exponents [Zoldi and Greenside, 1997b]. Thus, less unstable orbits are weighted more heavily. The approximation of each moment by orbits with period less than $\bar{T}$ has a form:

$$\tilde{M}_i(x,\bar{T}) = \frac{\sum\limits_{k:T_k<\bar{T}} \frac{1}{K_k}\frac{1}{T_k}\int_0^{T_k} x_k^i dt}{\sum\limits_{k:T_k<\bar{T}} \frac{1}{K_k}} \tag{12}$$

where $K_k$ is the positive Lyapunov exponent of the $k$th orbit and $T_k$ is its period.

We compare the moments calculated directly from a long time integration and approximated moments from all the periodic orbits with periods less than $\bar{T}$. The direct calculation has been performed for $10^7$ time units. This provides relative error of estimation of about 0.01%. As a quality of approximation we use the relative error of the approximation: $\dfrac{\tilde{M_i}(x,\bar{T}) - M_i(x)}{M_i(x)}$. The first 4 moments of the $z$ variable in the Lorenz model have been calculated directly and approximated. The relative error of the approximation as a function of $\bar{T}$, or the number of orbits used is shown in fig.5A. One can see the convergence of all moments to values calculated directly, however the convergence is rather slow, approximately as $1/\sqrt{N}$. To ameliorate convergence one can apply dynamical averaging [Cvitanovic, 1995] and the cycle expansion formalism [Arutso *et al.*, 1990a].

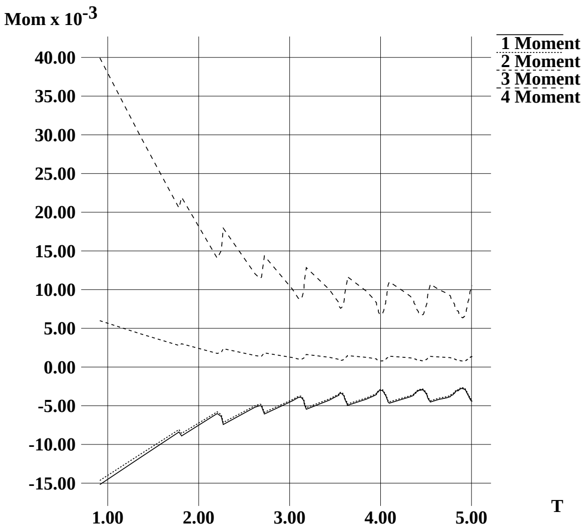


Figure 5A. Relative error of the approximation of average $z$ and its moments

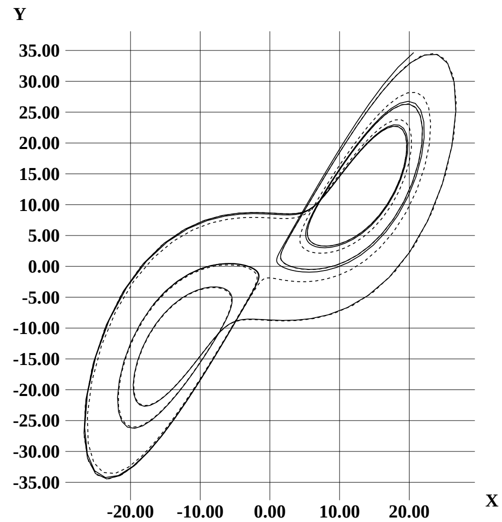


Figure 5B. Trajectory segment encoded by the symbol sequence "$BBBAABAAA$" and the periodic orbit with period $T = 4.045077$ with the same encoding.

## 4 Conclusion.

The application of periodic orbit formalism to the Lorenz model in this work and former ones cited in the text, points out the possibility and relative facility to characterise the model attractor properties through unstable periodic orbits. This fact rises the interest of performing the same type of analysis for a multi-dimensional PDE system like geophysical models. The numerical method used to calculate periodic orbits allows to find some low-period orbits for simple models of atmosphere and ocean like barotropic or multi-level quasi-geostrophic ones. Implementation of this method has another advantage because it requires the similar techniques as the data assimilation does, which is very well developed technique for this kind of models.

However, the transfer of these implications to geophysical models does not appear so straightforward. There exists a number of open questions which originates at principal differences between simple models, like Lorenz one, and PDE systems. Even if we know that there exist an unique solution and an attractor of a PDE system, the question of existence of periodic orbits should be studied carefully.

Moreover, the density of periodic solutions on the maximal attractor is even less evident. Hence it is not evident either the attractor set can be approximated by orbits, or some subset of the attractor only.

The orbits encoding, application of symbolic dynamics and possible cycle expansion allow to estimate easily many attractor properties and predictability characteristics. In particular, encoding of periodic orbits allows to know whether all the solutions with periods $T$ less than some $T_0$ have been found or not. For any missing orbit we can get a good initial guess for the descent procedure and thus find it. Application of the cycle expansion theory allows to obtain some attractor characteristics, which are difficult to calculate directly.

The encoding of orbits of high dimensional system obtained after discretisation of PDE system and application of the cycle expansion become much more difficult than encoding of the three-dimensional Lorenz model.

Nevertheless, the application of this methods, even partially, may give a powerful tool of the attractor and predictability studies for geophysical models.

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